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# Dispersive deformations of hydrodynamic reductions of (2 + 1)D dispersionless integrable systems

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## Abstract

We demonstrate that hydrodynamic reductions of dispersionless integrable systems in 2+1 dimensions, such as the dispersionless Kadomtsev–Petviashvili (dKP) and dispersionless Toda lattice (dTl) equations, can be deformed into reductions of the corresponding dispersive counterparts. Modulo the Miura group, such deformations are unique. The requirement that *any* hydrodynamic reduction possesses a deformation of this kind imposes strong constraints on the structure of dispersive terms, suggesting an alternative approach to the integrability in 2 + 1 dimensions.

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## 1. Introduction

The Kadomtsev–Petviashvili (KP) equation,

$$\left( u_t - uu_x - \frac{\epsilon^2}{12} u_{xxx} \right)_x = u_{yy}, \quad (1)$$

arises in mathematical physics as a two-dimensional generalization of the Korteweg-de-Vries (KdV) equation. Although its integrability aspects have been thoroughly investigated in the literature, we believe that one important property has been overlooked, namely, that equation (1) can be decoupled into a pair of consistent (1 + 1)-dimensional equations in a *continuum* number of ways. These decouplings can be obtained as deformations of hydrodynamic reductions of its dispersionless limit, known as the dKP equation,

$$(u_t - uu_x)_x = u_{yy}. \quad (2)$$

We point out that equation (2), also known as the Khokhlov–Zabolotskaya equation [32], is of interest in its own, arising in nonlinear acoustics, gas dynamics and differential geometry. A key property of the dKP equation is the existence of  $n$ -phase solutions of the form

$$u = u(R^1, \dots, R^n), \tag{3}$$

where the ‘phases’  $R^i(x, y, t)$  are governed by a pair of commuting hydrodynamic-type systems:

$$R_y^i = \mu^i(R)R_x^i, \quad R_t^i = \lambda^i(R)R_x^i. \tag{4}$$

Here  $\lambda^i = (\mu^i)^2 + u$ , while  $\mu^i$  and  $u$  satisfy the so-called Gibbons–Tsarev equations,

$$\partial_j \mu^i = \frac{\partial_j u}{\mu^j - \mu^i}, \quad \partial_i \partial_j u = 2 \frac{\partial_i u \partial_j u}{(\mu^j - \mu^i)^2}, \tag{5}$$

$i \neq j$ ,  $\partial_i = \partial/\partial R^i$ , which were first derived in [16, 17] in the context of hydrodynamic reductions of Benney’s moment equations. Thus, equations (4) can be viewed as a decomposition of the  $(2 + 1)$ -dimensional equation (2) into a pair of  $(1 + 1)$ -dimensional hydrodynamic-type systems. We will refer to equations (4) as hydrodynamic reductions of dKP. These reductions have been extensively studied in the literature; see, e.g., [9, 15, 23] and references therein. In particular, in the one-component case equations (5) become vacuous, and without any loss of generality one can set  $u(R) = R$  where  $R(x, y, t)$  satisfies a pair of Hopf-type equations:

$$R_y = \mu R_x, \quad R_t = (\mu^2 + R)R_x, \tag{6}$$

where  $\mu(R)$  is an arbitrary function. We recall that the general solution of equations (6) is given by the implicit formula  $f(R) = x + \mu y + (\mu^2 + R)t$ , which implies that the level surfaces  $R = \text{const}$  are planes. Solutions of this type are known as planar simple waves.

Our main observation is that *all* hydrodynamic reductions (4) can be deformed into reductions of the full KP equation by adding appropriate dispersive terms which are *polynomial* in the  $x$ -derivatives of  $R^i$ . Up to Miura-type transformations [5], such deformations are unique. Moreover, the calculation of dispersive corrections is an entirely algebraic procedure which does not require solving differential equations. In the one-component case one obtains the following deformation of equations (6):

$$R_y = \mu R_x + \frac{\epsilon^2}{12} \left( \mu' R_{xx} + \frac{1}{2} (\mu'' - (\mu')^3) R_x^2 \right) + O(\epsilon^4), \tag{7}$$

$$R_t = (\mu^2 + R)R_x + \frac{\epsilon^2}{12} \left( (2\mu\mu' + 1)R_{xx} + (\mu\mu'' - \mu(\mu')^3 + (\mu')^2/2)R_x^2 \right) + O(\epsilon^4).$$

See section 2.1 for more details. Note that the relation  $u = R$ , which now solves the full KP equation, remains undeformed: this can always be assumed modulo the Miura group. Equations (7) can be viewed as a decomposition of KP into a pair of commuting  $(1 + 1)$ -dimensional equations parametrized by an arbitrary function of one variable. In general, these equations constitute infinite series in  $\epsilon^2$  which can terminate only in exceptional cases (in the one-component situation this happens only when  $\mu = \text{const}$ , in which case KP reduces to KdV). One can show that deformations (7) are nontrivial, that is, not reducible to equations (6) by a Miura-type transformation. Formal expansions of the type (7) have appeared in [1], and were thoroughly investigated in a series of publications [5–8, 22, 29] in the context of 2D topological field theory. We would like to formulate the following conjecture:

*For any integrable soliton system in  $2 + 1$  dimensions, all hydrodynamic reductions of its dispersionless limit can be deformed into reductions of the original system.*

Thus, any  $(2 + 1)$ -dimensional integrable soliton equation can be decoupled into a pair of compatible  $(1 + 1)$ -dimensional equations in an infinite number of ways. Deformations of

two-component reductions of dKP are discussed in sections 2.2 and 2.3. Another example supporting our conjecture is provided by the Toda lattice system:

$$\begin{aligned} \epsilon u_y &= u(w(x) - w(x - \epsilon)), \\ \epsilon w_t &= u(x + \epsilon) - u(x), \end{aligned} \tag{8}$$

whose dispersionless limit (dTl equations) assumes the form

$$u_y = uw_x, \quad w_t = u_x.$$

In section 3, we provide explicit formulae for dispersive deformations of one-component reductions of dTl.

The requirement that *all* hydrodynamic reductions of a (2 + 1)-dimensional dispersionless integrable system can be deformed into reductions of the corresponding dispersive equation imposes strong constraints on the structure of dispersive terms. As an illustration, let us consider the generalized KP equation of the form

$$(u_t - uu_x + \epsilon(A_1u_{xx} + A_2u_x^2) + \epsilon^2(B_1u_{xxx} + B_2u_xu_{xx} + B_3u_x^3))_x = u_{yy}, \tag{9}$$

where  $A_i$  and  $B_i$  are certain functions of  $u$ . Note that this equation has the same dispersionless limit as the KP equation (1). As demonstrated in section 4.1, the requirement that *all* one-component reductions of dKP can be deformed into reductions of (9) readily implies  $A_1 = A_2 = B_2 = B_3 = 0$  and  $B_1 = \text{const}$ ; moreover, to establish this one only needs to perform calculations up to the order  $\epsilon^4$ . Thus, our procedure uniquely reconstructs the KP equation. Further examples of this type include BKP/CKP and the ‘universal’ equation (sections 4.2 and 4.3, respectively). This naturally leads to the program of classification of (2 + 1)-dimensional integrable soliton equations which can be summarized as follows:

- (a) Classify (2 + 1)-dimensional dispersionless integrable systems within various particularly interesting classes. A number of results in this direction are already available; see, e.g., [2, 9–14]. We recall that the integrability of a (2 + 1)-dimensional dispersionless system is understood as the existence, for any  $n$ , of an infinity of  $n$ -component hydrodynamic reductions parametrized by  $n$  arbitrary functions of one variable [9].
- (b) Reconstruct possible dispersive terms from the requirement that *all* hydrodynamic reductions of the dispersionless system can be deformed into reductions of the corresponding dispersive equation. We conjecture that any (2 + 1)-dimensional dispersionless integrable system can be deformed in this way (possibly, non-uniquely).

This scheme can be viewed as an alternative to the classical approach to the integrability in 2 + 1 dimensions, which starts with a linear dispersive part, say,  $u_{tx} - \frac{\epsilon^2}{12}u_{xxx} = u_{yy}$ , and reconstructs the allowed nonlinearity [24, 35].

## 2. Deformations of the dKP reductions

We will consider separately deformations of one- and two-component reductions of dKP.

### 2.1. One-component reductions

Let us first rewrite both KP and dKP equations as two-component systems:

$$u_t - uu_x - \frac{\epsilon^2}{12}u_{xxx} = w_y, \quad u_y = w_x \tag{10}$$

and

$$u_t - uu_x = w_y, \quad u_y = w_x, \tag{11}$$

respectively; such representation simplifies the calculations. One-component (one-phase) reductions of dKP are given by the formula

$$u = R, \quad w = w(R), \tag{12}$$

where  $w' = \mu$ , and  $R(x, y, t)$  solves equations (6). Dispersive deformations of equations (12), are sought in the form

$$\begin{aligned} u &= R, \\ w &= w(R) + \epsilon^2(b_1 R_{xx} + b_2 R_x^2) + \epsilon^4(d_1 R_{4x} + d_2 R_{xxx} R_x + d_3 R_{xx}^2 \\ &\quad + d_4 R_{xx} R_x^2 + d_5 R_x^4) + O(\epsilon^6), \end{aligned} \tag{13}$$

where the coefficients  $b_i$  and  $d_i$  are certain functions of  $R$ , and  $\partial_x^n R = R_{nx}$ . Note that the relation  $u = R$  remains undeformed: this can always be achieved modulo the Miura group. Similarly, the deformed version of equations (6) is

$$\begin{aligned} R_y &= \mu R_x + \epsilon^2(\varphi_1 R_{xxx} + \varphi_2 R_{xx} R_x + \varphi_3 R_x^3) + \epsilon^4(\rho_1 R_{5x} + \rho_2 R_{4x} R_x + \rho_3 R_{xxx} R_{xx} \\ &\quad + \rho_4 R_{xx}^2 R_x + \rho_5 R_{xxx} R_x^2 + \rho_6 R_{xx} R_x^3 + \rho_7 R_x^5) + O(\epsilon^6), \\ R_t &= (\mu^2 + R) R_x + \epsilon^2(\beta_1 R_{xxx} + \beta_2 R_{xx} R_x + \beta_3 R_x^3) + \epsilon^4(\delta_1 R_{5x} + \delta_2 R_{4x} R_x \\ &\quad + \delta_3 R_{xxx} R_{xx} + \delta_4 R_{xx}^2 R_x + \delta_5 R_{xxx} R_x^2 + \delta_6 R_{xx} R_x^3 + \delta_7 R_x^5) + O(\epsilon^6), \end{aligned} \tag{14}$$

where, again, coefficients are certain functions of  $R$  (it can be demonstrated that all odd order corrections in  $\epsilon$  must vanish identically). Substituting equations (13) into equations (10), using equations (14) and the compatibility condition  $R_{yt} = R_{ty}$ , we arrive at the recursive formulae for higher order corrections which are uniquely expressed in terms of  $\mu(R)$  and its derivatives. Thus, at the order  $\epsilon^2$  we obtain

$$\begin{aligned} b_1 &= \frac{\mu'}{12}, & b_2 &= \frac{1}{24}(\mu'' - (\mu')^3), \\ \varphi_1 &= \frac{\mu'}{12}, & \beta_1 &= \frac{1}{12}(2\mu\mu' + 1), \\ \varphi_2 &= \frac{1}{12}(2\mu'' - (\mu')^3), & \beta_2 &= \frac{1}{12}(4\mu\mu'' - 2\mu(\mu')^3 + 3(\mu')^2), \\ \varphi_3 &= \frac{1}{24}(\mu''' - 3(\mu')^2\mu''), & \beta_3 &= \frac{1}{12}(\mu\mu''' + 2\mu'\mu'' - 3\mu(\mu')^2\mu'' - (\mu')^4), \end{aligned}$$

which implies (7). Similarly, at the order  $\epsilon^4$  we have

$$\begin{aligned} d_1 &= \frac{1}{720}(3\mu'' - (\mu')^3), \\ d_2 &= \frac{1}{720}(6\mu''' - 19(\mu')^2\mu'' + (\mu')^5), \\ d_3 &= \frac{1}{1440}(9\mu''' - 16(\mu')^2\mu'' + 4(\mu')^5), \\ d_4 &= \frac{1}{1440}(11\mu^{(IV)} - 49(\mu')^2\mu''' - 84\mu'(\mu'')^2 + 52(\mu')^4\mu''), \\ d_5 &= \frac{1}{5760}(5\mu^{(V)} - 32(\mu')^2\mu^{(IV)} - 146\mu'\mu''\mu''' + 51(\mu')^4\mu''' - 44(\mu'')^3 + 216(\mu')^3(\mu'')^2). \end{aligned}$$

The remaining coefficients can be expressed as follows:

$$\begin{aligned} \rho_1 &= d_1, & \rho_2 &= d'_1 + d_2, & \rho_3 &= d_2 + 2d_3, \\ \rho_4 &= d'_3 + 2d_4, & \rho_5 &= d'_2 + d_4, & \rho_6 &= d'_4 + 4d_5, & \rho_7 &= d'_5, \end{aligned}$$

$$\begin{aligned}
 \delta_1 &= 2\mu d_1 + \frac{1}{144}(\mu')^2, \\
 \delta_2 &= 2\mu(d'_1 + d_2) + \frac{\mu'}{144}(720d_1 - 2(\mu')^3 + 5\mu''), \\
 \delta_3 &= 2\mu(d_2 + 2d_3) + \frac{\mu'}{144}(1440d_1 - 3(\mu')^3 + 8\mu''), \\
 \delta_4 &= 2\mu(d'_3 + 2d_4) + \frac{\mu''}{144}(2160d_1 + 4\mu'') \\
 &\quad + \frac{\mu'}{144}(432d_2 + 864d_3 - 22(\mu')^2\mu'' + 9\mu''' + (\mu')^5), \\
 \delta_5 &= 2\mu(d'_2 + d_4) + \frac{\mu''}{48}(480d_1 + \mu'') + \frac{\mu'}{144}(720d_2 - 16(\mu')^2\mu'' + 7\mu''' + (\mu')^5), \\
 \delta_6 &= 2\mu(d'_4 + 4d_5) + 2\mu''(d_3 + 3d_2) + \frac{5}{288}\mu'''(d_1 + 2\mu'') \\
 &\quad + \frac{\mu'}{288}(1440d_4 + 18(\mu')^4\mu'' - 78\mu'(\mu'')^2 - 35(\mu')^2\mu''' + 11\mu^{(IV)}), \\
 \delta_7 &= 2\mu d'_5 + \mu''d_4 + \frac{\mu^{(IV)}}{576}(d_1 + 2\mu'') + \frac{\mu'''}{576}(576d_2 + \mu''') \\
 &\quad + \frac{\mu'}{576}(2304d_5 + 21(\mu')^3(\mu'')^2 + 6(\mu')^4\mu''' - 48\mu'\mu''\mu''') \\
 &\quad - 8(\mu')^2\mu^{(IV)} - 24(\mu'')^3 + 2\mu^{(V)}).
 \end{aligned}$$

Although we have calculated dispersive corrections up to the order  $\epsilon^6$ , the expressions are getting increasingly more complicated. It is important to emphasize that the calculation of dispersive corrections is an entirely *algebraic* procedure which does not require solving differential equations.

## 2.2. Two-component reductions

Two-component reductions of the dKP equation (11) are sought in the form  $u = u(R^1, R^2)$ ,  $w = w(R^1, R^2)$ , where the Riemann invariants  $R^1$  and  $R^2$  satisfy the two-component version of equations (4). One has  $\lambda^i = (\mu^i)^2 + u$ ,  $\partial_j w = \mu^j \partial_j u$ , where  $\mu^j$  and  $u$  satisfy the two-component Gibbons–Tsarev system (5):

$$\partial_2 \mu^1 = \frac{\partial_2 u}{\mu^2 - \mu^1}, \quad \partial_1 \mu^2 = \frac{\partial_1 u}{\mu^1 - \mu^2}, \quad \partial_{12} u = \frac{2\partial_1 u \partial_2 u}{(\mu^2 - \mu^1)^2}. \quad (15)$$

We consider (1+1)-dimensional decompositions of the KP equation (1) obtained by deforming equations (6):

$$\begin{aligned}
 R_y^i &= \mu^i R_x^i + \epsilon^2 (a_j^i R_{xxx}^j + b_{jk}^i R_{xx}^j R_x^k + c_{jkl}^i R_x^j R_x^k R_x^l) + O(\epsilon^4), \\
 R_t^i &= \lambda^i R_x^i + \epsilon^2 (A_j^i R_{xxx}^j + B_{jk}^i R_{xx}^j R_x^k + C_{jkl}^i R_x^j R_x^k R_x^l) + O(\epsilon^4),
 \end{aligned} \quad (16)$$

where the coefficients  $a_j^i, A_j^i$ , etc, are certain functions of  $R^1, R^2$ ; here  $i, j, k, l = 1, 2$ . We point out that  $u(R^1, R^2)$  and  $w(R^1, R^2)$  remain undeformed (this can always be achieved modulo the Miura group). Moreover, one can show that all terms at the odd powers of  $\epsilon$  must vanish identically. Substituting  $u(R^1, R^2)$  and  $w(R^1, R^2)$  into equations (10), using equations (16) and imposing the compatibility condition  $R_{yt}^i = R_{ty}^i$ , one uniquely recovers

expressions for the coefficients in terms of  $\mu^i$  and  $u$ . Although these expressions are quite complicated in general, some of them are remarkably simple. Thus,

$$a_1^1 = \frac{1}{12} \frac{\partial_1 \mu^1}{\partial_1 u}, \quad a_2^1 = -\frac{1}{12} \frac{\partial_2 \mu^2}{\partial_1 u}, \quad a_1^2 = -\frac{1}{12} \frac{\partial_1 \mu^1}{\partial_2 u}, \quad a_2^2 = \frac{1}{12} \frac{\partial_2 \mu^2}{\partial_2 u},$$

or, in tensor notation,  $a_j^i = \frac{1}{12} (-1)^{i+j} \partial_j \mu^j / \partial_i u$ . Similarly,

$$A_1^1 = \frac{1}{12} + \frac{\mu^1 \partial_1 \mu^1}{6 \partial_1 u}, \quad A_2^1 = -\frac{(\mu^1 + \mu^2) \partial_2 \mu^2}{12 \partial_1 u},$$

$$A_1^2 = -\frac{(\mu^1 + \mu^2) \partial_1 \mu^1}{12 \partial_2 u}, \quad A_2^2 = \frac{1}{12} + \frac{\mu^2 \partial_2 \mu^2}{6 \partial_2 u},$$

or

$$A_j^i = \frac{1}{12} ((-1)^{i+j} (\mu^i + \mu^j) \partial_j \mu^j / \partial_i u + \delta_j^i).$$

As a particular case, let us consider deformations (16) such that the series truncates at the order  $\epsilon^2$ . This leads to the following two possibilities:

*Case 1.*

$$\partial_1 u = (\mu^2 - \mu^1) \partial_1 \mu^1, \quad \partial_2 u = (\mu^1 - \mu^2) \partial_2 \mu^2. \tag{17}$$

Equations (17) together with the Gibbons–Tsarev system (15) imply

$$\mu^1 + \mu^2 = \text{const.}$$

Setting  $\mu^1 = a + p$ ,  $\mu^2 = a - p$  where  $a = \text{const}$ , and substituting this representation into the first two equations in (15), one obtains  $u = b - p^2$ ,  $b = \text{const}$ . Then equation (15) (third equation) implies  $\partial_{12}(p^3) = 0$ , so that, up to a reparametrization of Riemann invariants, one has  $p = (R^1 - R^2)^{1/3}$ . Ultimately,

$$\mu^1 = a + (R^1 - R^2)^{1/3}, \quad \mu^2 = a + (R^2 - R^1)^{1/3}.$$

Up to a linear transformation of the independent variables, one can set  $a = b = 0$ . This implies  $\lambda^i = 0$  and  $R_i^i = 0$ , so that the system (16) (second equation) becomes trivial, while the system (16) (first equation) reduces to the stationary Boussinesq reduction of the dKP equation,  $(uu_x + \frac{\epsilon^2}{12} u_{xxx})_x + u_{yy} = 0$ .

*Case 2.*

$$\partial_1 u = \frac{1}{3} (\mu^1 - \mu^2) \partial_1 \mu^1, \quad \partial_2 u = \frac{1}{3} (\mu^2 - \mu^1) \partial_2 \mu^2. \tag{18}$$

Equations (18) together with the Gibbons–Tsarev system (15) imply that, up to a reparametrization of Riemann invariants, one can set

$$\mu^1 = \frac{3}{4} R^1 + \frac{1}{4} R^2, \quad \mu^2 = \frac{3}{4} R^2 + \frac{1}{4} R^1, \quad u = \frac{1}{16} (R^1 - R^2)^2.$$

The corresponding  $\epsilon^2$ -coefficients in equations (16) take the form

$$a_1^1 = a_1^2 = \frac{1}{2(R^1 - R^2)}, \quad a_2^1 = a_2^2 = \frac{1}{2(R^2 - R^1)},$$

$$A_1^1 = \frac{5R^1 + R^2}{6(R^1 - R^2)}, \quad A_2^1 = \frac{R^1 + R^2}{2(R^2 - R^1)},$$

$$A_1^2 = \frac{R^1 + R^2}{2(R^1 - R^2)}, \quad A_2^2 = \frac{R^1 + 5R^2}{6(R^2 - R^1)},$$

etc. Introducing the new dependent variables  $v = (R^1 + R^2)/2$  and  $u = (R^1 - R^2)^2/16$ , one can rewrite equations (16) as

$$u_y = (uv)_x, \quad v_y = vv_x + u_x + \frac{\epsilon^2}{4} \left( \frac{u_{xx}}{u} - \frac{1}{2} \frac{u_x^2}{u^2} \right)_x,$$

and

$$u_t = (2u + v^2)u_x + 2uvv_x + \frac{\epsilon^2}{12} \left( 4u_{xx} - 3 \frac{u_x^2}{u} \right)_x,$$

$$v_t = 2vv_x + (2u + v^2)v_x + \frac{\epsilon^2}{2} \left[ v \left( \frac{u_{xx}}{u} - \frac{1}{2} \frac{u_x^2}{u^2} \right) + \frac{1}{3} \left( 2v_{xx} + 3 \frac{u_x}{x} v_x \right) \right]_x,$$

respectively. This is the well-known Zakharov reduction of KP to NLS [33].

### 2.3. Waterbag reduction

It was observed in [26] that  $n$ -component reductions of dKP can also be sought in the form  $u = u(v^1, v^2, \dots, v^n)$ ,  $w = w(v^1, v^2, \dots, v^n)$ , where the fields  $v^i$  satisfy the equations

$$v_y^i = \left( \frac{(v^i)^2}{2} + u \right)_x, \quad v_t^i = \left( \frac{(v^i)^3}{3} + uv^i + w \right)_x. \quad (19)$$

The system (19) is automatically compatible provided  $u$  and  $w$  solve equations (11). The substitution into equation (11) (second equation) implies

$$\partial_i w = \left( \sum_{k=1}^n \partial_k u + v^i \right) \partial_i u, \quad (20)$$

where  $\partial_i = \partial_{v^i}$ . It turns out that equation (11) (first equation) is satisfied identically modulo equation (20). The consistency conditions  $\partial_i \partial_j w = \partial_j \partial_i w$  imply [26]

$$\partial_{ij} u = \frac{(\sum_{k \neq j} \partial_{ki} u) \partial_j u + (\sum_{k \neq i} \partial_{kj} u) \partial_i u}{\partial_i u - \partial_j u + v^i - v^j}, \quad (21)$$

which can be viewed as an analogue of the Gibbons–Tsarev system (5). In the two-component case,  $n = 2$ , the system (20) simplifies to

$$\partial_1 w = (\partial_1 u + \partial_2 u + v^1) \partial_1 u, \quad \partial_2 w = (\partial_1 u + \partial_2 u + v^2) \partial_2 u, \quad (22)$$

and the compatibility condition  $\partial_1 \partial_2 w = \partial_2 \partial_1 w$  takes the form

$$\partial_{12} u = \frac{\partial_2 u \partial_{11} u - \partial_1 u \partial_{22} u}{\partial_1 u - \partial_2 u + v^1 - v^2}. \quad (23)$$

This is an analogue of equations (15). The special case

$$u = \alpha_1 v^1 + \alpha_2 v^2, \quad w = \frac{\alpha_1 (v^1)^2 + \alpha_2 (v^2)^2}{2} + (\alpha_1 + \alpha_2)(\alpha_1 v^1 + \alpha_2 v^2), \quad (24)$$

where  $\alpha_1$  and  $\alpha_2$  are constants, is known as the *waterbag* reduction [18]. We seek a deformation of the two-component waterbag reduction (19) in the form

$$v_y^i = \left( \frac{(v^i)^2}{2} + u \right)_x + \epsilon^2 P^i + O(\epsilon^4),$$

$$v_t^i = \left( \frac{(v^i)^3}{3} + uv^i + w \right)_x + \epsilon^2 Q^i + O(\epsilon^4), \quad (25)$$

$i = 1, 2$ . Modulo the Miura group, we assume that equations (24) remain undeformed. Substituting equations (24) and (25) into (11) and using the compatibility conditions  $v_{yt}^i = v_{ty}^i$ ,



we obtain explicit expressions for  $P^i$  and  $Q^i$ . Thus, equations (25) (first equation) take the form

$$\begin{aligned} v_y^1 &= \left( \frac{(v^1)^2}{2} + u \right)_x + \frac{\epsilon^2}{\alpha_1} P + O(\epsilon^4), \\ v_y^2 &= \left( \frac{(v^2)^2}{2} + u \right)_x - \frac{\epsilon^2}{\alpha_2} P + O(\epsilon^4), \end{aligned} \tag{26}$$

where

$$\begin{aligned} P &= \frac{\Delta + 2\alpha_1 - \alpha_2}{12\Delta} v_{xxx}^1 - \frac{\Delta + \alpha_1 - 2\alpha_2}{12\Delta} v_{xxx}^2 + \frac{m - \alpha_2\Delta(\Delta - \alpha_1 - 2\alpha_2)}{12\alpha_1\alpha_2\Delta^2} v_{xx}^1 v_x^1 \\ &\quad - \frac{n\Delta + m}{12\alpha_1\alpha_2\Delta^2} (v_{xx}^1 v_x^2 + v_{xx}^2 v_x^1) + \frac{m + \alpha_1\Delta(\Delta + 2\alpha_1 + \alpha_2)}{12\alpha_1\alpha_2\Delta^2} v_{xx}^2 v_x^2 - \frac{a_2\Delta^2 + m}{24\alpha_1\alpha_2\Delta^3} \\ &\quad \times (v_x^1)^3 + \frac{\alpha_2\Delta^2 + 3m}{24\alpha_1\alpha_2\Delta^3} (v_x^1)^2 v_x^2 + \frac{\alpha_1\Delta^2 - 3m}{24\alpha_1\alpha_2\Delta^3} v_x^1 (v_x^2)^2 - \frac{\alpha_1\Delta^2 - m}{24\alpha_1\alpha_2\Delta^3} (v_x^2)^3, \end{aligned}$$

where

$$\Delta = v^1 - v^2, \quad m = \alpha_1^2(\alpha_1 - 2\alpha_2) - \alpha_2^2(\alpha_2 - 2\alpha_1), \quad n = \alpha_1^2 + \alpha_1\alpha_2 + \alpha_2^2.$$

Note that, although the hydrodynamic part of equations (26) is conservative, the expression  $P$  is not a total  $x$ -derivative. The expressions for  $Q^i$  are not presented here due to their complexity.

### 3. Deformations of the dTl reductions

Expanding the rhs in equations (8) one obtains

$$\begin{aligned} u_y/u &= w_x - \frac{\epsilon}{2} w_{xx} + \frac{\epsilon^2}{6} w_{xxx} + \dots + (-1)^{n+1} \frac{\epsilon^n}{n!} w_{nx} + \dots, \\ w_t &= u_x + \frac{\epsilon}{2} u_{xx} + \frac{\epsilon^2}{6} u_{xxx} + \dots + \frac{\epsilon^n}{n!} u_{nx} + \dots. \end{aligned} \tag{27}$$

The corresponding dispersionless limit, the dTl equation, is

$$u_y = uw_x, \quad w_t = u_x. \tag{28}$$

It admits one-component reductions of the form

$$u = R, \quad w = w(R),$$

where  $w' = 1/\mu$ , and  $R(x, y, t)$  satisfies a pair of Hopf-type equations:

$$R_t = \mu R_x, \quad R_y = \frac{R}{\mu} R_x, \tag{29}$$

where  $\mu(R)$  is an arbitrary function. Deformations are sought in the form

$$\begin{aligned} u &= R, \\ w &= w(R) + \epsilon a R_x + \epsilon^2 (b_1 R_{xx} + b_2 R_x^2) + \epsilon^3 (c_1 R_{xxx} + c_2 R_{xx} R_x + c_3 R_x^3) + O(\epsilon^4), \end{aligned} \tag{30}$$

where  $a, b_i, c_i$  are certain functions of  $R$ , and

$$\begin{aligned} R_t &= \mu R_x + \epsilon^2 (\beta_1 R_{xxx} + \beta_2 R_{xx} R_x + \beta_3 R_x^3) + O(\epsilon^4), \\ R_y &= \frac{R}{\mu} R_x + \epsilon^2 (\varphi_1 R_{xxx} + \varphi_2 R_{xx} R_x + \varphi_3 R_x^3) + O(\epsilon^4). \end{aligned} \tag{31}$$

We point out that the relation  $u = R$  remains undeformed (modulo the Miura group), and that all odd order corrections in  $\epsilon$  in equations (31) vanish identically. Substituting (30) into (27),

using (31) and the compatibility condition  $R_{yt} = R_{ty}$ , one can recursively calculate all higher order corrections in terms of  $\mu$  and its derivatives at different powers of  $\epsilon$ :

order  $\epsilon$ :

$$a = \frac{1}{2\mu},$$

order  $\epsilon^2$ :

$$b_1 = -\frac{R\mu' - 2\mu}{12\mu^2},$$

$$b_2 = -\frac{R\mu^2\mu'' - R^2(\mu')^3 + 3\mu^2\mu'}{24\mu^4},$$

$$\beta_1 = \frac{R\mu'}{12},$$

$$\beta_2 = \frac{R}{12\mu^2}((\mu')^2(\mu - R\mu') + 2\mu^2\mu''),$$

$$\beta_3 = \frac{R}{24\mu^3}(2R(\mu')^4 + 2\mu^2\mu'\mu'' - \mu(\mu')^2(2\mu' + 3R\mu'') + \mu^3\mu'''),$$

$$\varphi_1 = -\frac{R}{12\mu^2}(R\mu' - \mu),$$

$$\varphi_2 = \frac{R}{12\mu^4}(2R\mu(\mu')^2 + R^2(\mu')^3 - \mu^2(3\mu' + 2R\mu'')),$$

$$\varphi_3 = -\frac{R}{24\mu^5}(4R^2(\mu')^4 - 2\mu^2\mu'(R\mu'' + \mu') - R\mu(\mu')^2(3R\mu'' + 2\mu') + \mu^3(R\mu''' + 2\mu'')),$$

order  $\epsilon^3$ :

$$c_1 = -\frac{R\mu' - \mu}{24\mu^2},$$

$$c_2 = \frac{R^2(\mu')^3 + 2R\mu(\mu')^2 - \mu^2(2R\mu'' + 3\mu')}{24\mu^4}$$

$$c_3 = -\frac{1}{48\mu^5}(4R^2(\mu')^4 - 2\mu^2\mu'(R\mu'' + \mu') - R\mu(\mu')^2(3R\mu'' + 2\mu') + \mu^3(R\mu''' + 2\mu'')),$$

etc. Again, this calculation is an entirely algebraic procedure.

#### 4. Reconstruction of dispersive terms in (2 + 1)D

Given a dispersionless integrable system in 2 + 1 dimensions, how can one reconstruct the corresponding dispersive counterpart? This natural problem was first addressed by Zakharov in [34], based on the quantization of the corresponding dispersionless Lax pair. This, however, does not work when the Lax pair is ‘sufficiently complicated’. We propose an alternative approach to this problem, based on the requirement that *all* hydrodynamic reductions of the dispersionless system should be inherited by its dispersive counterpart. This imposes strong restrictions on the structure of dispersive terms. The following simple examples illustrate the method; further results in this direction will be reported elsewhere.

#### 4.1. Generalized KP equation

A two-component form of the generalized KP equation (9) is

$$\begin{aligned} u_t - uu_x + \epsilon(A_1u_{xx} + A_2u_x^2) + \epsilon^2(B_1u_{xxx} + B_2u_xu_{xx} + B_3u_x^3) &= w_y, \\ u_y &= w_x, \end{aligned} \quad (32)$$

where  $A_i$  and  $B_i$  are certain functions of  $u$ . Let us require that *any* one-component reduction (6) of its dispersionless limit can be deformed, in the form (14), into a reduction of equations (32). Thus, whenever one obtains a relation involving  $\mu(R)$  and its derivatives, all coefficients must be set equal to zero: this ensures that  $\mu(R)$  remains arbitrary. Looking at different powers of  $\epsilon$  we arrive at the following results:

order  $\epsilon$ :

$$A_1 = 0, \quad A_2 = 0,$$

order  $\epsilon^2$ :

$$B_1 = \text{const}, \quad B_3 = \frac{B_2'}{2},$$

order  $\epsilon^4$ :

$$3B_1B_2' - 2B_2^2 = 0.$$

This implies either  $B_2 = 0$ , or  $B_2 = -3B_1/(2u + c_0)$ , where  $c_0$  is an arbitrary integration constant. It follows that only the case

$$B_2 = 0$$

is admissible. Thus, the KP equation is the only possible extension, within the class (32), which inherits deformations of all hydrodynamic reductions.

#### 4.2. BKP and CKP equations

Let us consider equations of the form

$$\begin{aligned} u_t - 5(u^2 + w)u_x - 5uw_x + 5w_y + \epsilon^2(A_0uu_{xxx} + A_1w_{xxx} + A_2u_{xxx}) + \epsilon^4A_3u_{xxxxx} &= 0, \\ u_y &= w_x, \end{aligned} \quad (33)$$

where  $A_0$ – $A_3$  are arbitrary constants. Deformed one-component reductions are sought in the form

$$u = R,$$

$$\begin{aligned} w = w(R) + \epsilon^2(b_1R_{xx} + b_2R_x^2) + \epsilon^4(d_1R_{4x} + d_2R_{xxx}R_x + d_3R_{xx}^2 \\ + d_4R_{xx}R_x^2 + d_5R_x^4) + O(\epsilon^6), \end{aligned} \quad (34)$$

where  $R$  obeys the system of the form

$$\begin{aligned} R_y = \mu R_x + \epsilon^2(\varphi_1R_{xxx} + \varphi_2R_{xx}R_x + \varphi_3R_x^3) + \epsilon^4(\rho_1R_{5x} + \rho_2R_{4x}R_x + \rho_3R_{xxx}R_{xx} \\ + \rho_4R_{xx}^2R_x + \rho_5R_{xxx}R_x^2 + \rho_6R_{xx}R_x^3 + \rho_7R_x^5) + O(\epsilon^6), \\ R_t = 5(R^2 + w + \mu(R - \mu))R_x + \epsilon^2(\beta_1R_{xxx} + \beta_2R_{xx}R_x + \beta_3R_x^2) + \epsilon^4(\delta_1R_{5x} + \delta_2R_{4x}R_x \\ + \delta_3R_{xxx}R_{xx} + \delta_4R_{xx}^2R_x + \delta_5R_{xxx}R_x^2 + \delta_6R_{xx}R_x^3 + \delta_7R_x^5) + O(\epsilon^6), \end{aligned} \quad (35)$$

where  $\mu(R)$  is an arbitrary function, and  $w' = \mu$ . Substituting equations (34) and (35) into equations (33), we obtain  $\varphi, \rho, \beta, \delta$  in terms of  $b, d$ , etc. The compatibility condition  $R_{ty} = R_{yt}$  provides the following constraints on the constants  $A_i$ 's:

order  $\epsilon^2$ :

$$A_1 = A_0,$$

order  $\epsilon^4$ :

$$A_3 = -\frac{A_0^2}{25}, \quad 5A_0^2 - 7A_0A_2 + 2A_2^2 = 0.$$

The quadratic equation leads to either  $A_2 = A_0$  or  $A_2 = \frac{5}{2}A_0$ , which correspond to the BKP and CKP equations, respectively. We have verified that, up to the order  $\epsilon^6$ , no extra constraints appear, and that all coefficients in expansions (34) and (35) are certain expressions in terms of  $\mu$ . This calculation shows that the reconstruction of dispersive corrections is essentially non-unique.

### 4.3. The 'universal' equation

In this section, we study dispersive deformations of the so-called 'universal' equation,

$$\begin{aligned} u_y &= w_x, \\ u_t &= w_y + uw_x - wu_x, \end{aligned} \tag{36}$$

which was discussed in a series of publications [4, 25, 27]. The structure of hydrodynamic reductions in this case is particularly simple; namely,  $n$ -phase solutions are given by the formulae

$$u = R^1 + \dots + R^n, \quad w = f^1(R^1) + \dots + f^n(R^n) - \frac{1}{2}(R^1 + \dots + R^n)^2,$$

where  $f^i(R^i)$  are  $n$  arbitrary functions of one variable, and the phases  $R^i$  satisfy equations (4) with

$$\mu^i = f^{i'} - (R^1 + \dots + R^n), \quad \lambda^i = (\mu^i)^2 + u\mu^i - w.$$

Note that  $\mu^i = \partial_i w$ . In the one-component case, one has  $u = R, w = w(R)$ , where the single phase  $R$  satisfies the equations

$$R_y = \mu(R)R_x, \quad R_t = (\mu^2 + R\mu - w)R_x, \tag{37}$$

$w' = \mu$ . We have found the following dispersive deformation of equation (36):

$$\begin{aligned} u_y &= w_x, \\ u_t &= w_y + uw_x - wu_x + \tau\epsilon^4(u_{xx}w_{xxx} - w_{xx}u_{xxx}) + O(\epsilon^8), \end{aligned} \tag{38}$$

which inherits deformations of all one-components reductions (37) up to the order  $\epsilon^8$ . Our procedure, however, does not work beyond this order, leading to the trivial condition  $\tau = 0$ . We would like to emphasize that the extension of two-component reductions leads to the condition  $\tau = 0$  already at the order  $\epsilon^4$ . The deformation (38) is found by seeking dispersive extensions of equation (36) in the form

$$\begin{aligned} u_y &= w_x, \\ u_t &= w_y + uw_x - wu_x + A\epsilon(uw_{xx} - wu_{xx}) \\ &\quad + B_1\epsilon^2(uw_{xxx} - wu_{xxx}) + B_2\epsilon^2(u_xw_{xx} - w_xu_{xx}) \\ &\quad + C_1\epsilon^3(uw_{4x} - wu_{4x}) + C_2\epsilon^3(u_xw_{xxx} - w_xu_{xxx}) \\ &\quad + D_1\epsilon^4(uw_{5x} - wu_{5x}) + D_2\epsilon^4(u_xw_{4x} - w_xu_{4x}) + D_3\epsilon^4(u_{xx}w_{xxx} - w_{xx}u_{xxx}) \\ &\quad + E_1\epsilon^5(uw_{6x} - wu_{6x}) + E_2\epsilon^5(u_xw_{5x} - w_xu_{5x}) \\ &\quad + E_3\epsilon^5(u_{xx}w_{4x} - w_{xx}u_{4x}) + \dots, \end{aligned} \tag{39}$$

where the coefficients  $A, B_i$ , etc are constants. Note that one can simplify equation (39) by using Miura-type transformations of the form

$$u \rightarrow u + \alpha \epsilon u_x + \beta \epsilon^2 u_{xx} + \dots, \quad w \rightarrow w + \alpha \epsilon w_x + \beta \epsilon^2 w_{xx} + \dots,$$

where  $\alpha, \beta, \dots$  are arbitrary constants. In particular, using this freedom one can set  $B_2 = C_2 = D_2 = E_2 = 0$ , etc. In what follows, we assume this normalization. Let us require that any one-component reduction (37) can be extended as

$$\begin{aligned} u &= R, \\ w &= w(R) + \epsilon a R_x + \epsilon^2 (b_1 R_{xx} + b_2 R_x^2) + \epsilon^3 (c_1 R_{xxx} + c_2 R_x R_{xx} + c_3 R_x^3) \\ &\quad + \epsilon^4 (d_1 R_{4x} + d_2 R_{3x} R_x + d_3 R_{xx}^2 + d_4 R_{xx} R_x^2 + d_5 R_x^4) + \dots, \end{aligned} \tag{40}$$

where  $R$  satisfies the system

$$\begin{aligned} R_y &= \mu R_x + \epsilon (\sigma_1 R_{xx} + \sigma_2 R_x^2) + \epsilon^2 (\varphi_1 R_{xxx} + \varphi_2 R_{xx} R_x + \varphi_3 R_x^3) \\ &\quad + \epsilon^3 (\psi_1 R_{4x} + \psi_2 R_x R_{xxx} + \psi_3 R_{xx}^2 + \psi_4 R_{xx} R_x^2 + \psi_5 R_x^4) \\ &\quad + \epsilon^4 (\rho_1 R_{5x} + \rho_2 R_{4x} R_x + \rho_3 R_{3x} R_{xx} + \rho_4 R_{xx}^2 R_x \\ &\quad + \rho_5 R_{3x} R_x^2 + \rho_6 R_{xx} R_x^3 + \rho_7 R_x^5) + \dots, \\ R_t &= (\mu^2 + R\mu - w) R_x + \epsilon (\alpha_1 R_{xx} + \alpha_2 R_x^2) + \epsilon^2 (\beta_1 R_{xxx} + \beta_2 R_{xx} R_x + \beta_3 R_x^3) \\ &\quad + \epsilon^3 (\gamma_1 R_{4x} + \gamma_2 R_x R_{xxx} + \gamma_3 R_{xx}^2 + \gamma_4 R_{xx} R_x^2 + \gamma_5 R_x^4) + \psi_4 R_{xx} R_x^2 + \psi_5 R_x^4 \\ &\quad + \epsilon^4 (\delta_1 R_{5x} + \delta_2 R_{4x} R_x + \delta_3 R_{3x} R_{xx} + \delta_4 R_{xx}^2 R_x + \delta_5 R_{3x} R_x^2 + \delta_6 R_{xx} R_x^3 + \delta_7 R_x^5) \\ &\quad + \dots, \end{aligned} \tag{41}$$

$w' = \mu$ . Substituting (40) into (39), using equations (41) and the compatibility conditions  $R_{yt} = R_{ty}$  (where  $\mu$  is assumed to be arbitrary), at different orders in  $\epsilon$  we get the following conditions for the coefficients of (39):

order  $\epsilon$ :

$$A = 0,$$

order  $\epsilon^2$ :

$$B_1 = 0,$$

order  $\epsilon^3$ :

$$C_1 = 0,$$

order  $\epsilon^4$ :

$$D_1 = 0,$$

order  $\epsilon^5$ :

$$E_1 = E_3 = 0,$$

etc. Note that there is no condition on the coefficient  $D_3 = \tau$ . This results in equation (38). The corresponding equations (40), (41) take the form

$$\begin{aligned} u &= R, \\ w &= w(R) + \epsilon^4 (d_1 R_{4x} + d_2 R_{3x} R_x + d_3 R_{xx}^2 + d_4 R_{xx} R_x^2 + d_5 R_x^4) + O(\epsilon^8), \end{aligned}$$

and

$$R_y = \mu R_x + \epsilon^4 (\rho_1 R_{5x} + \rho_2 R_{4x} R_x + \rho_3 R_{3x} R_{xx} + \rho_4 R_{xx}^2 R_x + \rho_5 R_{3x} R_x^2 + \rho_6 R_{xx} R_x^3 + \rho_7 R_x^5) + O(\epsilon^8),$$

$$R_t = (\mu^2 + R\mu - w)R_x + \epsilon^4 (\delta_1 R_{5x} + \delta_2 R_{4x} R_x + \delta_3 R_{3x} R_{xx} + \delta_4 R_{xx}^2 R_x + \delta_5 R_{3x} R_x^2 + \delta_6 R_{xx} R_x^3 + \delta_7 R_x^5) + O(\epsilon^8),$$

respectively. Here all coefficients are certain explicit expressions in terms of  $\mu$  and its derivatives:

$$d_1 = 0, \quad d_2 = \frac{\tau \mu''}{3\mu'' - 2}, \quad d_3 = -\frac{3\tau \mu''}{3\mu'' - 2}, \quad d_4 = \frac{\tau(3\mu'' + 4)\mu'''}{3(\mu'')^2 - 11\mu'' + 6},$$

$$d_5 = \frac{\tau((74 - 19\mu'' - 12(\mu'')^2)(\mu''')^2 + (6 - 23\mu'' - 17(\mu'')^2 + 8(\mu'')^3)\mu^{(IV)})}{8(\mu'' - 3)^2(2 - 5\mu'' + 3(\mu'')^2)},$$

$$\delta_1 = 0, \quad \delta_2 = (R + 2\mu')\rho_2, \quad \delta_3 = (R + 2\mu')\rho_3,$$

$$\delta_4 = (R + 2\mu')\rho_4 + 3\mu''\rho_3 + 3\tau\mu''' - d_3,$$

$$\delta_5 = (R + 2\mu')\rho_5 + 5\mu''d_2 - \tau\mu''' - d_2,$$

$$\delta_6 = (R + 2\mu')\rho_6 + 2\mu'''(3d_2 + d_3) + \tau\mu'''' + 5\mu''d_4 - d_4,$$

$$\delta_7 = (R + 2\mu')\rho_7 + \mu^{(IV)}d_2 + \mu''d_4 + 4\mu''d_5 - d_5,$$

$$\rho_1 = 0, \quad \rho_2 = d_2, \quad \rho_3 = d_2 + 3d_3,$$

$$\rho_4 = d_3' + 2d_4, \quad \rho_5 = d_2' + d_4, \quad \rho_6 = d_4' + 4d_5, \quad \rho_7 = d_5'.$$

**Remark.** Reference [4] provides a multi-parameter deformation of the universal hierarchy with the first few terms given by

$$u_y = w_x,$$

$$u_t = w_y + uw_x - wu_x - \frac{\epsilon^2 \mu^2}{2\lambda^4} (u_{xx}w_{xxx} - w_{xx}u_{xxx}) + \frac{\epsilon^3 \mu^2}{2\lambda^5} (u_{xx}w_{4x} - w_{xx}u_{4x}) + \frac{\epsilon^3 \mu^3}{3\lambda^6} (u_{xxx}w_{4x} - w_{xxx}u_{4x}) - \frac{\epsilon^4 \mu^2}{2\lambda^6} (u_{xx}w_{5x} - w_{xx}u_{5x}) + \dots$$

Note that this deformation belongs to the class (39). According to our calculations, it *does not* inherit hydrodynamic reductions of the universal equation.

### 5. Concluding remarks

Several natural questions arise in the present context:

- (1) The results of [1, 5, 22] suggest that any evolution system in 1+1 dimensions is quasitriivial, that is, transformable to its dispersionless limit by a quasi-Miura transformation. In particular, deformed hydrodynamic reductions constructed in this paper are quasitriivial as well. It would be of interest to extend these quasitriiviality type results to the corresponding integrable systems in 2 + 1 dimensions such as the full KP and Toda lattice equations.
- (2) As demonstrated in [34], dispersionless limits of integrable systems in 2 + 1 dimensions come with Lax pairs of the form

$$\psi_y = A(\mathbf{u}, \psi_x), \quad \psi_t = B(\mathbf{u}, \psi_x), \tag{42}$$

which generate equations for the fields  $\mathbf{u}(x, y, t)$  via the compatibility conditions

$$A_t - B_y + \{A, B\} = 0,$$

where  $\{A, B\} = A_\xi B_x - B_\xi A_x$  denotes the standard Poisson bracket of  $A$  and  $B$ , and  $\xi = \psi_x$ . It was shown recently (see, e.g., [10, 12, 14]) that, conversely, for broad classes of multi-dimensional dispersionless systems, the existence of Lax pairs of the form (42) is necessary and sufficient for the integrability. In general, the dependence of  $A$  and  $B$  on  $\xi$  can be rather nontrivial, which makes direct ‘quantization’ of such Lax pairs difficult. Several ways to solve the problem of quantization of dispersionless Lax pairs of the form (42) were proposed in the literature, in particular, including the Moyal deformation of the Poisson bracket  $\{, \}$ ; see, e.g., [3, 4, 29, 30, 34]. These methods work well for a limited number of examples, allowing one to reconstruct the admissible dispersive terms; however, they meet difficulties when applied to Lax pairs with a more complicated dependence on  $\xi$ . Our approach to the problem of quantization is to allow only those dispersive corrections for which *all* hydrodynamic reductions of the dispersionless system can be deformed into reductions of its dispersive counterpart. We have demonstrated that this requirement is very restrictive indeed, and imposes strong constraints on the structure of possible dispersive terms. It would be challenging to apply this recipe to other classes of dispersionless integrable systems obtained in the literature, and to classify the associated soliton systems in  $2 + 1$  dimensions.

- (3) Our approach provides an infinity of decompositions of a given  $(2 + 1)$ -dimensional integrable soliton equation into a pair of commuting  $(1 + 1)$ -dimensional flows, which are parametrized by arbitrarily many functions of one variable. An alternative construction is known as the method of symmetry constraints, or potential–eigenfunction constraints; see, e.g., [19–21] and references therein. It would be of interest to understand whether generic deformed hydrodynamic reductions possess any kind of ‘symmetry’ interpretation.
- (4) Solutions of dispersionless systems coming from hydrodynamic reductions are known to break down in finite time. The addition of dispersive corrections can be viewed as a regularization procedure preventing the gradient catastrophe. Although the structure of higher order corrections can be rather complicated in general, it is the very first term that seems to be of prime importance. Thus, our procedure gives a canonical way to regularize hydrodynamic reductions. It would be of interest to investigate the behaviour of regularized solutions numerically.

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